The method of reflections and its applications to mean field models for sedimenting suspensions

Richard Höfer
Institut de Mathématiques de Bordeaux
Université de Bordeaux

Joint work with Juan J.L. Velázquez (Bonn), Richard Schubert (Aachen), Marta Leocata (Rome, LUISS) and Amina Mecherbet (Université de Paris)

Workshop *Collective behavior of particles in fluids*,
Institut Henri Poincaré, Paris, December 14, 2020

université de BORDEAUX

Institut de Mathématiques de Bordeaux

Outline

1. Introduction to the method of reflections
   - Historical comments
   - Different variants of the method
   - Convergence results

2. Applications to homogenization for particle suspensions
   - Derivation of Brinkman equations
   - Derivation of effective viscosity of suspensions

3. Application to mean field limits
   - Coupled transport-Stokes system and first order correction due to effective viscosity
   - Comments about Vlasov-Stokes(Brinkman) equations and models for non-spherical particles

First appearance of the method of reflections

Murphy 1833: describes a

"principle of successive influences"

in order to

"obtain numerical approximations to the state of electrised bodies influencing each other, by calculating the effects of 4 *or* 5 *successive acts of influence"*

First appearance of the method of reflections

Murphy 1833: describes a

"principle of successive influences"

in order to

"obtain numerical approximations to the state of electrised bodies influencing each other, by calculating the effects of 4 or 5 successive acts of influence"

Moreover, he writes:

"we may get the form for the state of B due to the influence of A, and then the state due to the influence of B or its own reflected influence"

First appearance of the method of reflections

Murphy 1833: describes a

"principle of successive influences"

in order to

"obtain numerical approximations to the state of electrised bodies influencing each other, by calculating the effects of 4 or 5 successive acts of influence"

Moreover, he writes:

"we may get the form for the state of B due to the influence of A, and then the state due to the influence of B or its own reflected influence"

He also gives the following explicit example: $A := B_1(x)$, $B := B_1(y)$, $x, y \in \mathbb{R}^3$, $|x - y| = 100$. Compute u (around A), where u solves

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{A \cup B}, \\
u &= \mathrm{const} \quad \text{in } \overline{A \cup B}, \\
1 &= \int_{\partial A} \nabla u \cdot n = \int_{\partial B} \nabla u \cdot n.
\end{aligned}
$$

Murphy's example

- Zero order approximation: $u_0 := \psi_0$,

B

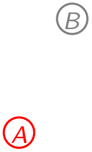
$$
\begin{aligned}
-\Delta \psi_0 &= 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{A}, \\
\psi_0 &= \mathrm{const} \quad \text{in } \overline{A}, \\
1 &= \int_{\partial A} \nabla \psi_0 \cdot n.
\end{aligned}
$$

A

Murphy's example

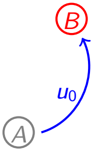
- Zero order approximation: $u_0 := \psi_0$,

$$-\Delta\psi_0 = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{A},$$
$$\psi_0 = \text{const} \quad \text{in } \overline{A},$$
$$1 = \int_{\partial A} \nabla\psi_0 \cdot n.$$



- First correction: $u_1 := u_0 + \psi_1$,

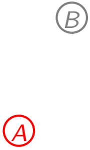
$$-\Delta\psi_1 = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{B},$$
$$\psi_1 = -u_0 + \text{const} \quad \text{in } \overline{B},$$
$$1 = \int_{\partial B} \nabla\psi_1 \cdot n.$$



Murphy's example

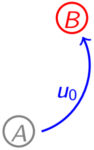
- Zero order approximation: $u_0 := \psi_0$,

$$\begin{aligned} -\Delta\psi_0 &= 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{A}, \\ \psi_0 &= \mathrm{const} \quad \text{in } \overline{A}, \\ 1 &= \int_{\partial A} \nabla\psi_0 \cdot n. \end{aligned}$$



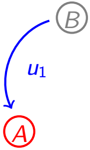
- First correction: $u_1 := u_0 + \psi_1$,

$$\begin{aligned} -\Delta\psi_1 &= 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{B}, \\ \psi_1 &= -u_0 + \mathrm{const} \quad \text{in } \overline{B}, \\ 1 &= \int_{\partial B} \nabla\psi_1 \cdot n. \end{aligned}$$



- Second correction: $u_2 := u_1 + \psi_2$,

$$\begin{aligned} -\Delta\psi_2 &= 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{A}, \\ \psi_2 &= -u_1 + \mathrm{const} \quad \text{in } \overline{A}, \\ 0 &= \int_{\partial A} \nabla\psi_2 \cdot n. \end{aligned}$$



Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.
- Described in many textbooks, e.g.: Happel–Brenner '65, Kim–Karrila '91, Dhont '96

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.
- Described in many textbooks, e.g.: Happel–Brenner '65, Kim–Karrila '91, Dhont '96
- First rigorous convergence results: Ozawa '83, Luke '89, Balabane '97, Traytak '06

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.
- Described in many textbooks, e.g.: Happel–Brenner '65, Kim–Karrila '91, Dhont '96
- First rigorous convergence results: Ozawa '83, Luke '89, Balabane '97, Traytak '06
- Further analysis and application to homogenization and mean field problems: Orlandi–Figari–Teta '85, Rubinstein '86, Jabin–Otto'04, H.-Velázquez '18, H. '18, Mecherbet '19, Hillairet–Lacave–Wu '19, H. '19, Hillairet–Wu '20, Gérard-Varet–Hillairet '20, Gérard-Varet–Mecherbet '20 H.–Schubert '20

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.
- Described in many textbooks, e.g.: Happel–Brenner '65, Kim–Karrila '91, Dhont '96
- First rigorous convergence results: Ozawa '83, Luke '89, Balabane '97, Traytak '06
- Further analysis and application to homogenization and mean field problems: Orlandi–Figari–Teta '85, Rubinstein '86, Jabin–Otto'04, H.-Velázquez '18, H. '18, Mecherbet '19, Hillairet–Lacave–Wu '19, H. '19, Hillairet–Wu '20, Gérard-Varet–Hillairet '20, Gérard-Varet–Mecherbet '20 H.–Schubert '20
- Investigated as a tool in numerical analysis: Laurent–Legendre–Salomon '17, Ciaramella–Gander–Halpern–Salomon '19

Further development of the method

- Further mentioning of the method in the context of particles immersed in a fluid by Lamb 1906, Lorentz 1907, Smoluchowski 1911.
- Smoluchowski 1911: Detailed description of the method for 2 and N particles with given velocities; formal criterium for convergence: $NR/L \ll 1$.
- Golusin 1938: the method is always converging for $N = 2$ but not for $N > 2$.
- Described in many textbooks, e.g.: Happel–Brenner '65, Kim–Karrila '91, Dhont '96
- First rigorous convergence results: Ozawa '83, Luke '89, Balabane '97, Traytak '06
- Further analysis and application to homogenization and mean field problems: Orlandi–Figari–Teta '85, Rubinstein '86, Jabin–Otto'04, H.-Velázquez '18, H. '18, Mecherbet '19, Hillairet–Lacave–Wu '19, H. '19, Hillairet–Wu '20, Gérard-Varet–Hillairet '20, Gérard-Varet–Mecherbet '20 H.–Schubert '20
- Investigated as a tool in numerical analysis: Laurent–Legendre–Salomon '17, Ciaramella–Gander–Halpern–Salomon '19
- Nice review: Ciaramella–Gander–Halpern–Salomon '17

Formulation in terms of orthogonal projections

Let I finite or countable, $B_i := B_R(X_i)$. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ solve

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i}$$

$$u = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i}.$$

Formulation in terms of orthogonal projections

Let I finite or countable, $B_i := B_R(X_i)$. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ solve

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i}$$

$$u = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i}.$$

Consider the spaces

$$V_i := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \overline{B}_i \},$$
$$V := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \bigcup_{i \in I} \overline{B}_i \} = \bigcap_{i \in I} V_i.$$

Let $v = St^{-1}_{\mathbb{R}^3} f$. Then

$$u = Pv, \qquad P \colon \dot{H}^1_\sigma(\mathbb{R}^3) \to V \quad \text{orthogonal projection.}$$

Formulation in terms of orthogonal projections

Let I finite or countable, $B_i := B_R(X_i)$. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ solve

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i}$$
$$u = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i}.$$

Consider the spaces

$$V_i := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \overline{B}_i \},$$
$$V := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \bigcup_{i \in I} \overline{B}_i \} = \bigcap_{i \in I} V_i.$$

Let $v = St^{-1}_{\mathbb{R}^3} f$. Then

$$u = Pv, \qquad P \colon \dot{H}^1_\sigma(\mathbb{R}^3) \to V \quad \text{orthogonal projection.}$$

On the other hand, $\varphi := Q_i \psi := (1 - P_i)\psi$ solves

$$-\Delta \varphi + \nabla p = 0 \quad \operatorname{div} \varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

Formulation in terms of orthogonal projections

Let I finite or countable, $B_i := B_R(X_i)$. Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ solve

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i}$$

$$u = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i}.$$

Consider the spaces

$$V_i := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \overline{B}_i \},$$
$$V := \{ w \in \dot{H}^1_\sigma(\mathbb{R}^3) : w = 0 \text{ in } \bigcup_{i \in I} \overline{B}_i \} = \bigcap_{i \in I} V_i.$$

Let $v = St^{-1}_{\mathbb{R}^3} f$. Then

$$u = Pv, \qquad P \colon \dot{H}^1_\sigma(\mathbb{R}^3) \to V \quad \text{orthogonal projection.}$$

On the other hand, $\varphi := Q_i \psi := (1 - P_i)\psi$ solves

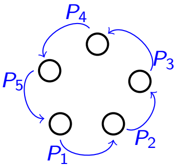
$$-\Delta \varphi + \nabla p = 0 \quad \operatorname{div} \varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

If $|I| = 2$, the method of reflections as described by Murphy is just expressed by

$$u_k := (P_2 P_1)^k v$$

The sequential, the parallel and the averaged parallel version

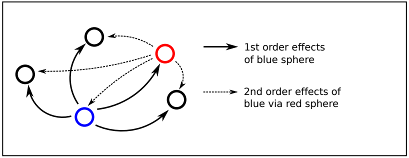
sequential: $u_k^{(s)} := (P_N P_{N-1} \dots P_1)^k v$



The sequential, the parallel and the averaged parallel version

sequential: $u_k^{(s)} := (P_N P_{N-1} \dots P_1)^k v$

parallel: $u_k^{(p)} := \left(1 - \sum_{l=1}^{N} Q_l\right)^k v$

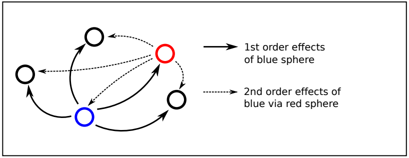


$$\left(1 - \sum_{l} Q_l\right)^k = 1 - \sum_{l_1} Q_{l_1} + \sum_{l_1} \sum_{l_2 \neq l_1} Q_{l_1} Q_{l_2} - \dots + (-1)^k \sum_{l_1} \dots \sum_{l_k \neq l_{k-1}} Q_{l_1} \dots Q_{l_k}$$

The sequential, the parallel and the averaged parallel version

sequential: $u_k^{(s)} := (P_N P_{N-1} \dots P_1)^k v$

parallel: $u_k^{(p)} := \left(1 - \sum_{l=1}^{N} Q_l\right)^k v$



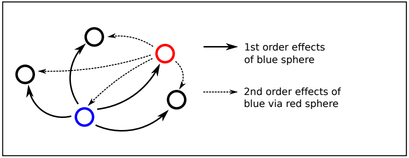
$$\left(1 - \sum_l Q_l\right)^k = 1 - \sum_{l_1} Q_{l_1} + \sum_{l_1} \sum_{l_2 \neq l_1} Q_{l_1} Q_{l_2} - \dots + (-1)^k \sum_{l_1} \dots \sum_{l_k \neq l_{k-1}} Q_{l_1} \dots Q_{l_k}$$

- The sequential method is always converging ([Luke '89])

The sequential, the parallel and the averaged parallel version

sequential: $u_k^{(s)} := (P_N P_{N-1} \ldots P_1)^k v$

parallel: $u_k^{(p)} := \left(1 - \sum_{l=1}^{N} Q_l\right)^k v$



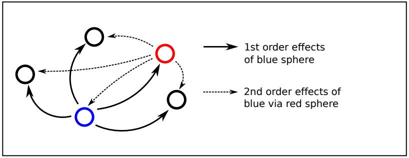
$$\left(1 - \sum_{l} Q_l\right)^k = 1 - \sum_{l_1} Q_{l_1} + \sum_{l_1} \sum_{l_2 \neq l_1} Q_{l_1} Q_{l_2} - \cdots + (-1)^k \sum_{l_1} \cdots \sum_{l_k \neq l_{k-1}} Q_{l_1} \ldots Q_{l_k}$$

- The sequential method is always converging ([Luke '89])
- The parallel method is much more practical: all particles treated the same, separation of order of interactions and scales.

The sequential, the parallel and the averaged parallel version

sequential: $u_k^{(s)} := (P_N P_{N-1} \dots P_1)^k v$

parallel: $u_k^{(p)} := \left(1 - \sum_{l=1}^{N} Q_l\right)^k v$



$$\left(1 - \sum_l Q_l\right)^k = 1 - \sum_{l_1} Q_{l_1} + \sum_{l_1} \sum_{l_2 \neq l_1} Q_{l_1} Q_{l_2} - \dots + (-1)^k \sum_{l_1} \dots \sum_{l_k \neq l_{k-1}} Q_{l_1} \dots Q_{l_k}$$

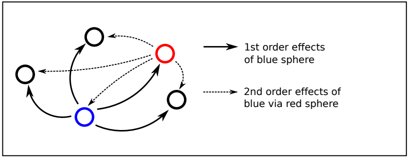
- The sequential method is always converging ([Luke '89])
- The parallel method is much more practical: all particles treated the same, separation of order of interactions and scales.
- The parallel method is converging iff $L := \sum_l Q_l$ satisfies $\|L\| < 2$.
- If $L \geq c > 0$ on $V^{\perp} = \ker L$, then for $\delta > 0$

$$\|P - (1 - \delta L)^k\| \leq (\max\{1 - \delta c, \delta \|L\| - 1\})^k .$$

The sequential, the parallel and the averaged parallel version

sequential: $u_k^{(s)} := (P_N P_{N-1} \dots P_1)^k v$

parallel: $u_k^{(p)} := \left(1 - \sum_{l=1}^{N} Q_l\right)^k v$



$$\left(1 - \sum_l Q_l\right)^k = 1 - \sum_{l_1} Q_{l_1} + \sum_{l_1} \sum_{l_2 \neq l_1} Q_{l_1} Q_{l_2} - \dots + (-1)^k \sum_{l_1} \dots \sum_{l_k \neq l_{k-1}} Q_{l_1} \dots Q_{l_k}$$

- The sequential method is always converging ([Luke '89])
- The parallel method is much more practical: all particles treated the same, separation of order of interactions and scales.
- The parallel method is converging iff $L := \sum_l Q_l$ satisfies $\|L\| < 2$.
- If $L \geq c > 0$ on $V^\perp = \ker L$, then for $\delta > 0$

$$\|P - (1 - \delta L)^k\| \leq (\max\{1 - \delta c, \delta\|L\| - 1\})^k .$$

- If $N = |I| < \infty$, the averaged parallel version with $\delta = \frac{1}{N}$ is always converging.

Convergence results for Dirichlet boundary conditions

Recall: $\varphi := Q_i \psi$ solves

$$-\Delta \varphi + \nabla p = 0 \quad \operatorname{div} \varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

$$(Q_i \psi)(x) = 6\pi R \psi(X_i) \Phi(x - X_i) + O\left(\frac{R^2}{|x|^2}\right), \qquad \Phi(x) = \frac{1}{8\pi}\left(\frac{1}{|x|} + \frac{x \otimes x}{|x|^3}\right).$$

Convergence results for Dirichlet boundary conditions

Recall: $\varphi := Q_i \psi$ solves

$$-\Delta \varphi + \nabla p = 0 \quad \operatorname{div} \varphi = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

$$(Q_i \psi)(x) = 6\pi R \psi(X_i) \Phi(x - X_i) + O\left(\frac{R^2}{|x|^2}\right), \qquad \Phi(x) = \frac{1}{8\pi}\left(\frac{1}{|x|} + \frac{x \otimes x}{|x|^3}\right).$$

Therefore, in B_j : $\qquad |(1 - L)\psi)| = |\sum_{i \neq j} Q_i \psi| \leq C \sum_{i \neq j} \frac{R\psi(X_i)}{|X_i - X_j|}.$

Convergence results for Dirichlet boundary conditions

Recall: $\varphi := Q_i\psi$ solves

$$-\Delta\varphi + \nabla p = 0 \quad \operatorname{div}\varphi = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

$$(Q_i\psi)(x) = 6\pi R\psi(X_i)\Phi(x - X_i) + O\left(\frac{R^2}{|x|^2}\right), \qquad \Phi(x) = \frac{1}{8\pi}\left(\frac{1}{|x|} + \frac{x \otimes x}{|x|^3}\right).$$

$$\text{Therefore, in } B_j: \qquad |(1-L)\psi)| = |\sum_{i\neq j} Q_i\psi| \leq C \sum_{i\neq j} \frac{R\psi(X_i)}{|X_i - X_j|}.$$

- [Jabin–Otto '04, Traytak '06]: Using a maximum (modulus) principle, this leads to convergence of the method in L^∞ if $d_{\min} \geq 4R$ and

$$\max_i \sum_{j\neq i} \frac{R}{|X_i - X_j|} \leq \varepsilon_0.$$

- H^1-setting: [H.–Velázquez '18]

Convergence results for Dirichlet boundary conditions

Recall: $\varphi := Q_i \psi$ solves

$$-\Delta \varphi + \nabla p = 0 \quad \operatorname{div} \varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

$$(Q_i \psi)(x) = 6\pi R \psi(X_i) \Phi(x - X_i) + O\left(\frac{R^2}{|x|^2}\right), \qquad \Phi(x) = \frac{1}{8\pi}\left(\frac{1}{|x|} + \frac{x \otimes x}{|x|^3}\right).$$

$$\text{Therefore, in } B_j : \qquad |(1 - L)\psi)| = |\sum_{i \neq j} Q_i \psi| \leq C \sum_{i \neq j} \frac{R\psi(X_i)}{|X_i - X_j|}.$$

- [Jabin–Otto '04, Traytak '06]: Using a maximum (modulus) principle, this leads to convergence of the method in L^∞ if $d_{\min} \geq 4R$ and

$$\max_i \sum_{j \neq i} \frac{R}{|X_i - X_j|} \leq \varepsilon_0.$$

- H^1-setting: [H.–Velázquez '18]
- If the particles satisfy $|X_i - X_j| \geq D N^{-1/3}$, then the condition simplifies to $NR/D \ll 1$ as stated by Smoluchowski 1911.

Convergence results for Dirichlet boundary conditions

Recall: $\varphi := Q_i\psi$ solves

$$-\Delta\varphi + \nabla p = 0 \quad \operatorname{div}\varphi = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{B_i}$$
$$\varphi = \psi \quad \text{in } \overline{B_i}.$$

$$(Q_i\psi)(x) = 6\pi R\psi(X_i)\Phi(x - X_i) + O\left(\frac{R^2}{|x|^2}\right), \qquad \Phi(x) = \frac{1}{8\pi}\left(\frac{1}{|x|} + \frac{x\otimes x}{|x|^3}\right).$$

$$\text{Therefore, in } B_j: \qquad |(1-L)\psi)| = |\sum_{i\neq j} Q_i\psi| \leq C\sum_{i\neq j}\frac{R\psi(X_i)}{|X_i - X_j|}.$$

- [Jabin–Otto '04, Traytak '06]: Using a maximum (modulus) principle, this leads to convergence of the method in L^∞ if $d_{\min} \geq 4R$ and

$$\max_i \sum_{j\neq i}\frac{R}{|X_i - X_j|} \leq \varepsilon_0.$$

- H^1-setting: [H.–Velázquez '18]
- If the particles satisfy $|X_i - X_j| \geq DN^{-1/3}$, then the condition simplifies to $NR/D \ll 1$ as stated by Smoluchowski 1911.
- Ozawa '83: Convergence of an approximate version in a random setting.

Convergence results for sedimentation boundary conditions

Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ be the solution to

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \backslash \bigcup_{i \in I} \overline{B_i},$$

$$Du = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i},$$

$$\int_{\partial B_i} \sigma[u,p]n = \int_{B_i} f, \quad \int_{\partial B_i} \sigma[u,p]n \times (x - X_i) = \int_{B_i} f \times (x - X_i).$$

Convergence results for sedimentation boundary conditions

Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ be the solution to

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i},$$

$$Du = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i},$$

$$\int_{\partial B_i} \sigma[u,p]n = \int_{B_i} f, \quad \int_{\partial B_i} \sigma[u,p]n \times (x - X_i) = \int_{B_i} f \times (x - X_i).$$

In this case

$$V := \{ w \in \dot{H}^1(\mathbb{R}^3) : Dw = 0 \text{ in } \bigcup_{i \in I} \overline{B_i} \}$$

and $\varphi = Q_i \psi$ solves

$$-\Delta \varphi + \nabla q = 0 \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i},$$

$$D\varphi = D\psi \quad \text{in } \bigcup_{i \in I} \overline{B_i},$$

$$0 = \int_{\partial B_i} \sigma[u,p]n = \int_{\partial B_i} \sigma[u,p]n \times (x - X_i).$$

Convergence results for sedimentation boundary conditions

Let $f \in \dot{H}^{-1}(\mathbb{R}^3)$ and let $u \in \dot{H}^1(\mathbb{R}^3)$ be the solution to

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i},$$

$$Du = 0 \quad \text{in } \bigcup_{i \in I} \overline{B_i},$$

$$\int_{\partial B_i} \sigma[u,p]n = \int_{B_i} f, \quad \int_{\partial B_i} \sigma[u,p]n \times (x - X_i) = \int_{B_i} f \times (x - X_i).$$

In this case

$$V := \{ w \in \dot{H}^1(\mathbb{R}^3) : Dw = 0 \text{ in } \bigcup_{i \in I} \overline{B_i} \}$$

and $\varphi = Q_i \psi$ solves

$$-\Delta \varphi + \nabla q = 0 \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i \in I} \overline{B_i},$$

$$D\varphi = D\psi \quad \text{in } \bigcup_{i \in I} \overline{B_i},$$

$$0 = \int_{\partial B_i} \sigma[u,p]n = \int_{\partial B_i} \sigma[u,p]n \times (x - X_i).$$

This leads to

$$(Q_i \psi)(x) = 5|B_1|R^3 D\psi(x_i) \nabla \Phi(x - X_i) + O\left(\frac{R^4}{|x^3|}\right).$$

Better convergence properties due to dipoles

Since

$$(Q_i\psi)(x) = R^3 D\psi(x_i)\nabla\Phi(x - X_i) + O\left(\frac{R^4}{|x^3|}\right).$$

we have in B_j

$$|D(1 - L)\psi)| = |\sum_{i \neq j} DQ_i\psi| \leq \sum_{i \neq j} \frac{R^3\psi(X_i)}{|X_i - X_j|^3}$$

Better convergence properties due to dipoles

Since

$$(Q_i\psi)(x) = R^3 D\psi(x_i)\nabla\Phi(x - X_i) + O\left(\frac{R^4}{|x^3|}\right).$$

we have in B_j

$$|D(1-L)\psi)| = |\sum_{i\neq j} DQ_i\psi| \leq \sum_{i\neq j} \frac{R^3\psi(X_i)}{|X_i - X_j|^3}$$

- [H. '18]: If

$$d_{\min} \geq cN^{-1/3}$$
$$\phi \log N := NR^3 \log N \leq \varepsilon_0,$$

the method converges in L^∞.

Better convergence properties due to dipoles

Since

$$(Q_i\psi)(x) = R^3 D\psi(x_i)\nabla\Phi(x - X_i) + O\left(\frac{R^4}{|x^3|}\right).$$

we have in B_j

$$|D(1-L)\psi)| = |\sum_{i\neq j} DQ_i\psi| \leq \sum_{i\neq j} \frac{R^3\psi(X_i)}{|X_i - X_j|^3}$$

- [H. '18]: If

$$d_{\min} \geq cN^{-1/3}$$
$$\phi \log N := NR^3 \log N \leq \varepsilon_0,$$

  the method converges in L^∞.
- [Hillairet–Wu '20]: Using that $\nabla^2\Phi$ is a Calderon-Zygmund operator, one can get rid of the logarithm in an L^p setting $1 < p < \infty$.

Better convergence properties due to dipoles

Since

$$(Q_i\psi)(x) = R^3 D\psi(x_i)\nabla\Phi(x - X_i) + O\left(\frac{R^4}{|x^3|}\right).$$

we have in B_j

$$|D(1 - L)\psi)| = |\sum_{i \neq j} DQ_i\psi| \leq \sum_{i \neq j} \frac{R^3\psi(X_i)}{|X_i - X_j|^3}$$

- [H. '18]: If

$$d_{\min} \geq cN^{-1/3}$$
$$\phi \log N := NR^3 \log N \leq \varepsilon_0,$$

  the method converges in L^∞.
- [Hillairet–Wu '20]: Using that $\nabla^2\Phi$ is a Calderon-Zygmund operator, one can get rid of the logarithm in an L^p setting $1 < p < \infty$.
- [H. '19]: Convergence of the method in all $\dot{W}^{1,p}$, $1 < p < \infty$ with rate ϕ.

Part II

Application of the method of reflections to homogenization

The Brinkman equations - formal considerations

For $R > 0$, let $X_i \in \mathbb{R}^3$, $i \in I$ and assume

$$\lim_{R \to 0} R \sum_{i \in I} \delta_{X_i} = \mu,$$

and let u_R be the solution to

$$-\Delta u_R + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{i \in I} \overline{B_i}$$
$$u_R = 0 \quad \text{in } \cup_{i \in I} \overline{B_i}.$$

The Brinkman equations - formal considerations

For $R > 0$, let $X_i \in \mathbb{R}^3$, $i \in I$ and assume

$$\lim_{R\to 0} R \sum_{i\in I} \delta_{X_i} = \mu,$$

and let u_R be the solution to

$$-\Delta u_R + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{i\in I} \overline{B_i}$$
$$u_R = 0 \quad \text{in } \cup_{i\in I} \overline{B_i}.$$

Formally, with δ_x^R the uniform measure on $B_R(x)$,

$$\begin{aligned}
u_R &= \mathrm{St}_{\mathbb{R}^3}^{-1} f - \sum_{i_1} Q_{i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} f + \sum_{i_1} \sum_{i_2 \neq i_1} Q_{i_1} Q_{i_2} \mathrm{St}_{\mathbb{R}^3}^{-1} f - \dots \\
&\approx \mathrm{St}_{\mathbb{R}^3}^{-1} f - \sum_{i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta_{X_{i_1}}^R \mathrm{St}_{\mathbb{R}^3}^{-1} f) + \sum_{i_1} \sum_{i_2 \neq i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta_{X_{i_1}}^R \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta_{X_{i_2}}^R \mathrm{St}_{\mathbb{R}^3}^{-1} f)) - \dots \\
&\to \mathrm{St}_{\mathbb{R}^3}^{-1} f - \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi\mu \mathrm{St}_{\mathbb{R}^3}^{-1} f) + \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi\mu \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi\mu \mathrm{St}_{\mathbb{R}^3}^{-1} f)) - \dots \\
&= \sum_{k=0}^{\infty} (-1)^k (\mathrm{St}_{\mathbb{R}^3}^{-1} 6\pi\mu)^k \mathrm{St}_{\mathbb{R}^3}^{-1} f \\
&= (\mathrm{St}_{\mathbb{R}^3} + 6\pi\mu)^{-1} f,
\end{aligned}$$

The Brinkman equations - formal considerations

For $R > 0$, let $X_i \in \mathbb{R}^3$, $i \in I$ and assume

$$\lim_{R\to 0} R \sum_{i\in I} \delta_{X_i} = \mu,$$

and let u_R be the solution to

$$-\Delta u_R + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{i\in I} \overline{B_i}$$
$$u_R = 0 \quad \text{in } \cup_{i\in I} \overline{B_i}.$$

Formally, with δ_x^R the uniform measure on $B_R(x)$,

$$\begin{aligned} u_R &= \mathrm{St}_{\mathbb{R}^3}^{-1} f - \sum_{i_1} Q_{i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} f + \sum_{i_1} \sum_{i_2 \neq i_1} Q_{i_1} Q_{i_2} \mathrm{St}_{\mathbb{R}^3}^{-1} f - \dots \\ &\approx \mathrm{St}_{\mathbb{R}^3}^{-1} f - \sum_{i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta^R_{X_{i_1}} \mathrm{St}_{\mathbb{R}^3}^{-1} f) + \sum_{i_1} \sum_{i_2 \neq i_1} \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta^R_{X_{i_1}} \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi R \delta^R_{X_{i_2}} \mathrm{St}_{\mathbb{R}^3}^{-1} f)) - \dots \\ &\to \mathrm{St}_{\mathbb{R}^3}^{-1} f - \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi \mu \mathrm{St}_{\mathbb{R}^3}^{-1} f) + \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi \mu \mathrm{St}_{\mathbb{R}^3}^{-1} (6\pi \mu \mathrm{St}_{\mathbb{R}^3}^{-1} f)) - \dots \\ &= \sum_{k=0}^{\infty} (-1)^k (\mathrm{St}_{\mathbb{R}^3}^{-1} 6\pi\mu)^k \mathrm{St}_{\mathbb{R}^3}^{-1} f \\ &= (\mathrm{St}_{\mathbb{R}^3} + 6\pi\mu)^{-1} f, \end{aligned}$$

where $u = (\mathrm{St}_{\mathbb{R}^3} + 6\pi\mu)^{-1} f$ has to be understood as

$$-\Delta u + \nabla p + 6\pi\mu u = 0, \quad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3.$$

The Brinkman equations - rigorous results

- Ozawa '83, Orlandi–Figari–Teta '85, Rubinstein '86: N i.i.d. particles with radius $1/N$, starting from the equations

$$-\Delta u_N + \nabla p_N + \lambda u = 0 \quad \text{in } \mathbb{R}^3 \backslash \bigcup_i B_i$$

with λ sufficiently large.

The Brinkman equations - rigorous results

- Ozawa '83, Orlandi–Figari–Teta '85, Rubinstein '86: N i.i.d. particles with radius $1/N$, starting from the equations

$$-\Delta u_N + \nabla p_N + \lambda u = 0 \quad \text{in } \mathbb{R}^3 \backslash \bigcup_i B_i$$

  with λ sufficiently large.

- H.-Velázquez '18: Well separated-particles in the whole of $\mathbb{R}^3$ without the massive term λu.
  We use a relaxed version of the method of reflections:

$$u_k := \left(1 - \gamma \sum_i e^{-|x_i|} Q_i\right)^k \mathrm{St}_{\mathbb{R}^3}^{-1} f.$$

The Brinkman equations - rigorous results

- Ozawa '83, Orlandi–Figari–Teta '85, Rubinstein '86: N i.i.d. particles with radius $1/N$, starting from the equations

$$-\Delta u_N + \nabla p_N + \lambda u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_i B_i$$

  with λ sufficiently large.

- H.-Velázquez '18: Well separated-particles in the whole of $\mathbb{R}^3$ without the massive term λu.
  We use a relaxed version of the method of reflections:

$$u_k := \left(1 - \gamma \sum_i e^{-|x_i|} Q_i\right)^k \mathrm{St}_{\mathbb{R}^3}^{-1} f.$$

- Further results (without using the method of reflections):
  Allaire '90, Desvillettes-Golse-Ricci '09, Feireisl-Namlyeyeva-Nečasová '16, Hillairet-Moussa-Sueur '19, Giunti-H. '19, Carrapatoso-Hillairet '20, H.-Jansen '20, ...

Einstein's formula for the effective viscosity

Recall that (formally) the solution $u \in \dot{H}^1(\mathbb{R}^3)$ to

$$-\Delta u + \nabla p = f \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \overline{B_i},$$

$$Du = 0 \quad \text{in } \bigcup_{i=1}^{N} \overline{B_i},$$

$$\int_{\partial B_i} \sigma[u,p]n = \int_{B_i} f,$$

$$\int_{\partial B_i} \sigma[u,p]n \times (x - X_i) = \int_{B_i} f \times (x - X_i).$$

is given by

$$u = u_1 + O(\phi^2) = \mathrm{St}_{\mathbb{R}^3}^{-1} f - \sum_i Q_i \mathrm{St}_{\mathbb{R}^3}^{-1} f + O(\phi^2).$$

Einstein's formula for the effective viscosity

Assume $\rho_N = \frac{1}{N}\sum_i \delta_{X_i} \to \rho$. Then with $v = \mathrm{St}_{\mathbb{R}^3}^{-1} f$,

$$\lim_{N\to\infty} u_1 = v - 5\phi \lim_{N\to\infty} \frac{1}{N}\sum_i Dv(X_i)\nabla\Phi(\cdot - X_i) = v - 5\phi\mathrm{St}_{\mathbb{R}^3}^{-1}(\operatorname{div}\rho Dv) =: w$$

Einstein's formula for the effective viscosity

Assume $\rho_N = \frac{1}{N}\sum_i \delta_{X_i} \to \rho$. Then with $v = \mathrm{St}_{\mathbb{R}^3}^{-1} f$,

$$\lim_{N\to\infty} u_1 = v - 5\phi \lim_{N\to\infty} \frac{1}{N}\sum_i Dv(X_i)\nabla\Phi(\cdot - X_i) = v - 5\phi \mathrm{St}_{\mathbb{R}^3}^{-1}(\operatorname{div} \rho Dv) =: w$$

The function w solves

$$-\operatorname{div}(2Dw + 5\phi\rho Dv) + \nabla p = f, \quad \operatorname{div} w = 0.$$

Since $w - v = O(\phi)$, we have $u_* - w = O(\phi^2)$, where

$$-\operatorname{div}((2 + 5\phi\rho) Du_*) + \nabla q = f, \quad \operatorname{div} u_* = 0.$$

Einstein's formula for the effective viscosity

Assume $\rho_N = \frac{1}{N}\sum_i \delta_{X_i} \to \rho$. Then with $v = \mathrm{St}_{\mathbb{R}^3}^{-1} f$,

$$\lim_{N\to\infty} u_1 = v - 5\phi \lim_{N\to\infty} \frac{1}{N}\sum_i Dv(X_i)\nabla\Phi(\cdot - X_i) = v - 5\phi \mathrm{St}_{\mathbb{R}^3}^{-1}(\operatorname{div} \rho Dv) =: w$$

The function w solves

$$-\operatorname{div}(2Dw + 5\phi\rho Dv) + \nabla p = f, \quad \operatorname{div} w = 0.$$

Since $w - v = O(\phi)$, we have $u_* - w = O(\phi^2)$, where

$$-\operatorname{div}((2 + 5\phi\rho)Du_*) + \nabla q = f, \quad \operatorname{div} u_* = 0.$$

- Rigorous results: Niethammer–Schubert '19, Hillairet–Wu '20
- Further results: Gérard-Varet–Hillairet '20, Gérard-Varet–Mecherbet '20 H.–Schubert '20, Gérard-Varet–H. '20, Gloria–Duerinckx '20, Gérard-Varet '20, Duerinckx '20

Part III

Application to mean field limits

Microscopic dynamics of inertialess particles in a Stokes flow

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} B_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \leq i \leq N$$

$$\int_{\partial B_i} \sigma[u_N, p_N] n \,\mathrm{d}\mathcal{H}^2 = \frac{g}{N}, \qquad \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \,\mathrm{d}\mathcal{H}^2 = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = V_i,$$

Microscopic dynamics of inertialess particles in a Stokes flow

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N$$

$$\int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2 = \frac{g}{N}, \qquad \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = V_i,$$

To fit into the previous framework, choose

$$f = \frac{g}{N} \sum_{i=1}^N \delta_{X_i}^R$$

Microscopic dynamics of inertialess particles in a Stokes flow

$$
\begin{aligned}
-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i, \\
u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N \\
\int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2 = \frac{g}{N}, \qquad \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = 0, \\
\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = V_i,
\end{aligned}
$$

To fit into the previous framework, choose

$$
f = \frac{g}{N} \sum_{i=1}^N \delta_{X_i}^R
$$

Recall: by Stokes law, the sedimentation velocity of a single particle is

$$
V_{\mathrm{St}} = \frac{g}{6\pi N R} =: \frac{g}{6\pi\gamma}.
$$

Thus, γ is the interaction strength.

Rigorous derivation of the transport-Stokes system

Jabin-Otto '04: If $\Lambda(0) := \frac{RN^{2/3}}{d_{\min}(0)} \leq \varepsilon_0$, then

$$|V_i - V_{\mathrm{St}}| \leq C\Lambda(0)|V_{\mathrm{St}}| \text{ for all } 1 \leq i \leq N.$$

Note that $d_{\min} \leq CN^{-1/3}$, thus $\Lambda(0) \geq \gamma = NR$.

Rigorous derivation of the transport-Stokes system

Jabin-Otto '04: If $\Lambda(0) := \frac{RN^{2/3}}{d_{\min}(0)} \leq \varepsilon_0$, then

$$|V_i - V_{\mathrm{St}}| \leq C\Lambda(0)|V_{\mathrm{St}}| \text{ for all } 1 \leq i \leq N.$$

Note that $d_{\min} \leq CN^{-1/3}$, thus $\Lambda(0) \geq \gamma = NR$.

H. '18: Assume $\rho_N(0) = \frac{1}{N}\sum_i \delta_{X_i(0)} \to \rho_0$ and

$$d_{\min} \geq cN^{-1/3}, \tag{A1}$$

$$\phi \log N := NR^3 \log N \to 0, \tag{A2}$$

$$NR \to \gamma \in (0, \infty]. \tag{A3}$$

Then $\rho_N(t) \to \rho(t)$ which solves the transport-Stokes system.

$$\partial_t \rho + \left(u + (6\pi\gamma)^{-1} g\right) \cdot \nabla_x \rho = 0, \qquad \rho(0) = \rho_0$$
$$-\Delta u + \nabla p = \rho g, \qquad \operatorname{div} u = 0.$$

Rigorous derivation of the transport-Stokes system

Jabin-Otto '04: If $\Lambda(0) := \frac{RN^{2/3}}{d_{\min}(0)} \leq \varepsilon_0$, then

$$|V_i - V_{\mathrm{St}}| \leq C\Lambda(0)|V_{\mathrm{St}}| \text{ for all } 1 \leq i \leq N.$$

Note that $d_{\min} \leq CN^{-1/3}$, thus $\Lambda(0) \geq \gamma = NR$.

H. '18: Assume $\rho_N(0) = \frac{1}{N}\sum_i \delta_{X_i(0)} \to \rho_0$ and

$$d_{\min} \geq cN^{-1/3}, \quad \text{(A1)}$$

$$\phi \log N := NR^3 \log N \to 0, \quad \text{(A2)}$$

$$NR \to \gamma \in (0,\infty]. \quad \text{(A3)}$$

Then $\rho_N(t) \to \rho(t)$ which solves the transport-Stokes system.

$$\partial_t \rho + \left(u + (6\pi\gamma)^{-1} g\right) \cdot \nabla_x \rho = 0, \qquad \rho(0) = \rho_0$$
$$-\Delta u + \nabla p = \rho g, \qquad \operatorname{div} u = 0.$$

Mecherbet '19: Relaxation of (A1) in the case $\gamma \leq \varepsilon_0$, quantitative convergence in Wasserstein distance, particle rotations included.

First order correction in the volume fraction ϕ_N due to an effective viscosity

Theorem (H. & Schubert '20)

Assume

$$d_{\min} \geq cN^{-\frac{1}{3}},$$
$$\phi_N \log N \to 0,$$
$$\mathcal{W}_\infty(\rho_N(0), \rho_0) = o(\phi_N) \quad \text{for some } \rho_0 \in W^{1,\infty}(\mathbb{R}^3).$$

where $\mathcal{W}_p$, $1 \leq p \leq \infty$ *denotes the Wasserstein distance.*
Let ρ *be the unique solution to*

$$\partial_t \rho + \left(u_{eff} + (6\pi NR)^{-1} g\right) \cdot \nabla \rho = 0, \quad \rho(0, \cdot) = \rho_0,$$
$$\operatorname{div}\left(2\left(1 + \frac{5}{2}\phi_N \rho\right) Du_{eff}\right) + \nabla p = \rho g, \quad \operatorname{div} u_{eff} = 0.$$

Then, for all $1 \leq p < \infty$ *and all* $T > 0$, *for all* N *sufficiently large and all* $t \leq T$

$$\mathcal{W}_p(\rho_N(t), \rho(t)) \leq C\left(\phi_N^2 |\log \phi_N| + \mathcal{W}_p(\rho_N(0), \rho(0))\right) e^{Ct}.$$

Moreover, for $q < 3$ *and* $p > \max\{1, \frac{3q}{3+q}\}$

$$\| u_N(t) - u_{eff} \|_{L^q_{\mathrm{loc}}} \leq C\left(\phi_N^2 |\log \phi_N| + \mathcal{W}_p(\rho_N(0), \rho(0))\right) e^{Ct}$$

First step: explicit approximation of the particle velocities

- Using the method of reflections,

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = u_N(X_i)$$

$$\approx \frac{g}{6\pi RN} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N \frac{1}{N^2} \sum_{j \neq i} \sum_{k \neq j} D\Phi(X_i - X_j) D\Phi(X_j - X_k) g$$

First step: explicit approximation of the particle velocities

- Using the method of reflections,

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) = u_N(X_i)$$
$$\approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N\frac{1}{N^2}\sum_{j\neq i}\sum_{k\neq j}D\Phi(X_i - X_j)D\Phi(X_j - X_k)g$$

- Need estimates for $u_N(X_i) - u_N(X_j)$ to control particle distances.

First step: explicit approximation of the particle velocities

- Using the method of reflections,

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) = u_N(X_i)$$

$$\approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N\frac{1}{N^2}\sum_{j\neq i}\sum_{k\neq j}D\Phi(X_i - X_j)D\Phi(X_j - X_k)g$$

- Need estimates for $u_N(X_i) - u_N(X_j)$ to control particle distances.
- Thus, we still need $\phi_N \log N \ll 1$.

Second step: adaptation of a result by Hauray

Let K satisfies $\mathrm{div} K = 0$, $|K| + |x||\nabla K| \leq C|x|^{-\alpha}$ for some $\alpha < d - 1$.

Theorem (Hauray '09)
Let

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j).$$

Let ρ be the solution to $\partial_t \rho + (K * \rho) \cdot \nabla \rho = 0$ with initial datum $\rho_0 \in L^\infty \cap \mathcal{P}$. Denote $W_\infty(t) = W_\infty(\rho_N(t), \rho(t))$. Then,

$$\frac{(W_\infty(0))^d}{(d_{\min}(0))^{1+\alpha}} \to 0 \quad \Rightarrow \quad \forall T > 0 \ \exists N_0 \in \mathbb{N} \ \forall N > N_0 \ \forall t \leq T \quad W_\infty(t) \leq W_\infty(0) e^{Ct}.$$

Second step: adaptation of a result by Hauray

Let K satisfies $\mathrm{div} K = 0$, $|K| + |x||\nabla K| \leq C|x|^{-\alpha}$ for some $\alpha < d - 1$.

Theorem (Hauray '09)

Let

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j).$$

Let ρ be the solution to $\partial_t \rho + (K * \rho) \cdot \nabla \rho = 0$ with initial datum $\rho_0 \in L^\infty \cap \mathcal{P}$. Denote $W_\infty(t) = W_\infty(\rho_N(t), \rho(t))$. Then,

$$\frac{(W_\infty(0))^d}{(d_{\min}(0))^{1+\alpha}} \to 0 \quad \Rightarrow \quad \forall T > 0 \; \exists N_0 \in \mathbb{N} \; \forall N > N_0 \; \forall t \leq T \quad W_\infty(t) \leq W_\infty(0) e^{Ct}.$$

Theorem (H.-Schubert '20 (rough statement))

Let

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j) + \phi_N(X_i) + E_i(t).$$

Assume $\mathrm{div}\, \phi_N = 0$ and let ρ be the solution to $\partial_t \rho + (K * \rho + \phi_N) \cdot \nabla \rho = 0$ with initial datum $\rho_0 \in L^\infty \cap \mathcal{P}$. Assume there exists a monotone function $e_N(t)$ such that

$$\forall \lambda > 0 \; \exists N_0 \in \mathbb{N} \; \forall N > N_0 \quad \frac{d_{\min}(0)}{d_{\min}(t)} + \frac{W_\infty(t)}{W_\infty(0) + e_N(t)} \leq \lambda \quad \Rightarrow \quad E_i(t) \leq e_N(t).$$

Then,

$$\forall t > 0 \; \frac{(W_\infty(0) + e_N(t))^{d-(1+\alpha)}}{(d_{\min}(0))^{1+\alpha} N^{(1+\alpha)/d}} \to 0 \;\Rightarrow\; \forall T > 0 \ldots \quad W_\infty(t) \leq C(W_\infty(0) + e_N(t)) e^{Ct}.$$

First application of the abstract theorem

Recall

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N\frac{1}{N^2}\sum_{j\neq i}\sum_{k\neq j}D\Phi(X_i - X_j)D\Phi(X_j - X_k)g$$

First application of the abstract theorem

Recall

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N \frac{1}{N^2}\sum_{j\neq i}\sum_{k\neq j} D\Phi(X_i - X_j)D\Phi(X_j - X_k)g$$

Treat last term as error to deduce

$$\mathcal{W}_\infty(\rho_N(t), \tau(t)) \leq C(\phi_N + W_\infty(\rho_N(0), \rho_0))e^{Ct}.$$

where τ is the solution to

$$\partial_t \tau + (v + (6\pi NR)^{-1} g)\cdot\nabla\tau = 0, \quad \tau(0,\cdot) = \rho_0,$$
$$-\Delta v + \nabla p = \tau g, \quad \mathrm{div}\, v = 0$$

First application of the abstract theorem

Recall

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N \frac{1}{N^2} \sum_{j \neq i} \sum_{k \neq j} D\Phi(X_i - X_j) D\Phi(X_j - X_k) g$$

Treat last term as error to deduce

$$\mathcal{W}_\infty(\rho_N(t), \tau(t)) \leq C(\phi_N + W_\infty(\rho_N(0), \rho_0)) e^{Ct}.$$

where τ is the solution to

$$\begin{aligned} \partial_t \tau + (v + (6\pi NR)^{-1} g) \cdot \nabla \tau = 0, \quad \tau(0, \cdot) = \rho_0, \\ -\Delta v + \nabla p = \tau g, \quad \operatorname{div} v = 0 \end{aligned}$$

Using this,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} X_i(t) &\approx \frac{g}{6\pi RN} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N D\Phi * (\tau(D\Phi g * \tau))(X_i) \\ &= \frac{g}{6\pi RN} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N D\Phi * (\tau D v)(X_i) \end{aligned}$$

Conclusion of the argument

We apply again the abstract theorem, this time to

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i \neq j} \Phi(X_i - X_j)g - 5\phi_N D\Phi * (\tau Dv)(X_i)$$

We observe that $5\phi_N D\Phi * (\tau Dv)(X_i)$ is the solution to the Stokes equations with source term $5\phi_N \operatorname{div}(\tau Dv)$.

Conclusion of the argument

We apply again the abstract theorem, this time to

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N D\Phi * (\tau Dv)(X_i)$$

We observe that $5\phi_N D\Phi * (\tau Dv)(X_i)$ is the solution to the Stokes equations with source term $5\phi_N \operatorname{div}(\tau Dv)$.

Thus we obtain as the mean field limit

$$\partial_t\tilde{\rho} + \left(\tilde{u} + (6\pi NR)^{-1}g\right)\cdot\nabla\tilde{\rho} = 0, \quad \tilde{\rho}(0,\cdot) = \rho_0,$$
$$\operatorname{div}\left(2D\tilde{u} + 5\phi_N\tau Dv\right) + \nabla p = \tilde{\rho}g, \quad \operatorname{div}\tilde{u} = 0.$$

Conclusion of the argument

We apply again the abstract theorem, this time to

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N D\Phi * (\tau Dv)(X_i)$$

We observe that $5\phi_N D\Phi * (\tau Dv)(X_i)$ is the solution to the Stokes equations with source term $5\phi_N \operatorname{div}(\tau Dv)$.

Thus we obtain as the mean field limit

$$\partial_t\tilde{\rho} + \left(\tilde{u} + (6\pi NR)^{-1}g\right)\cdot\nabla\tilde{\rho} = 0, \quad \tilde{\rho}(0,\cdot) = \rho_0,$$
$$\operatorname{div}\left(2D\tilde{u} + 5\phi_N\tau Dv\right) + \nabla p = \tilde{\rho}g, \quad \operatorname{div}\tilde{u} = 0.$$

Finally, we compare this to the desired limit system

$$\partial_t\rho + \left(u_{eff} + (6\pi Nr)^{-1}g\right)\cdot\nabla\rho = 0, \quad \rho(0,\cdot) = \rho_0,$$
$$\operatorname{div}\left(2Du_{eff} + 5\phi_N\rho Du_{eff}\right) + \nabla p = \rho g, \quad \operatorname{div}u_{eff} = 0.$$

and show that $\|\tilde{u} - u_{eff}\| + W_p(\tilde{\rho},\rho) \leq C\phi^2$.

Sedimentation of inertial particles – Vlasov-Stokes equations

For inertial particles, i.e.

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} B_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_i = g - \frac{\gamma}{R} \int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Omega_i = -\frac{\gamma}{R^3} \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2.$$

one formally obtains in the limit $N \to \infty$, $NR \to \gamma$ the Vlasov-Stokes system

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v (gf + 6\pi\gamma(u - v) f) = 0,$$

$$-\Delta u + \nabla p = 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, \mathrm{d}v, \quad \operatorname{div} u = 0,$$

Sedimentation of inertial particles – Vlasov-Stokes equations

For inertial particles, i.e.

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i,$$
$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} V_i = g - \frac{\gamma}{R} \int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \Omega_i = -\frac{\gamma}{R^3} \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2.$$

one formally obtains in the limit $N \to \infty$, $NR \to \gamma$ the Vlasov-Stokes system

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v \left(gf + 6\pi\gamma(u - v) f \right) = 0,$$
$$-\Delta u + \nabla p = 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, \mathrm{d}v, \quad \operatorname{div} u = 0,$$

- Difficulties: Second order; no chance to propagate bounds on $d_{\min}$; slow convergence of the method of reflections

Sedimentation of inertial particles – Vlasov-Stokes equations

For inertial particles, i.e.

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i,$$
$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \leq i \leq N,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} V_i = g - \frac{\gamma}{R} \int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \Omega_i = -\frac{\gamma}{R^3} \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2.$$

one formally obtains in the limit $N \to \infty$, $NR \to \gamma$ the Vlasov-Stokes system

$$\partial_t f + v \cdot \nabla_x f + \operatorname{div}_v (gf + 6\pi\gamma(u - v) f) = 0,$$
$$-\Delta u + \nabla p = 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, \mathrm{d}v, \quad \operatorname{div} u = 0,$$

- Difficulties: Second order; no chance to propagate bounds on $d_{\min}$; slow convergence of the method of reflections
- Derivation starting from different models: Bernard–Desvillettes–Golse–Ricci '18; Flandoli–Leocata–Ricci '19

Inertialess rodlike particles

For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f) = D_t \Delta_x f + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \mathrm{div}\, \sigma := \mathrm{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \mathrm{div}\, u = 0.$$

Analyzed by Helzel, Tzavaras and Otto.

Inertialess rodlike particles

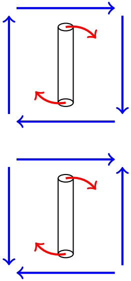
For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f) = D_t \Delta_x f + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \mathrm{div}\, \sigma := \mathrm{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \mathrm{div}\, u = 0.$$

Analyzed by Helzel, Tzavaras and Otto.

Ongoing work with Amina Mecherbet and Marta Leocata: Derive the onset of the stress σ from a microscopic model.



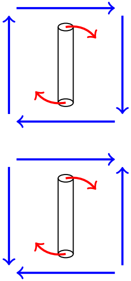
Inertialess rodlike particles

For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f) = D_t \Delta_x f + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \mathrm{div}\, \sigma := \mathrm{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \mathrm{div}\, u = 0.$$

Analyzed by Helzel, Tzavaras and Otto.



Ongoing work with Amina Mecherbet and Marta Leocata:
Derive the onset of the stress σ from a microscopic model.

$$-\Delta u_N + \nabla p_N = 0, \quad \mathrm{div}\, u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_i \mathcal{B}_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } \mathcal{B}_i,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \, \mathrm{d}\mathcal{H}^2 = \sqrt{\mathcal{R}_1(\xi_i)} \frac{\mathrm{d}\bar{B}_i}{\mathrm{d}t},$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = \sqrt{\mathcal{R}_2(\xi_i)} \frac{\mathrm{d}B_i}{\mathrm{d}t},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i = V_i, \quad \frac{\mathrm{d}}{\mathrm{d}t} \xi_i = \xi_i \times \Omega_i.$$

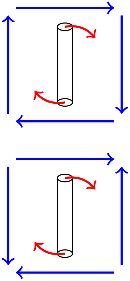
Inertialess rodlike particles

For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + \cancel{u \cdot \nabla_x f} + \operatorname{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f) = \cancel{D_t \Delta_x f} + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \operatorname{div} \sigma := \operatorname{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \operatorname{div} u = 0.$$

Analyzed by Helzel, Tzavaras and Otto.



Ongoing work with Amina Mecherbet and Marta Leocata:
Derive the onset of the stress σ from a microscopic model.

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_i \mathcal{B}_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } \mathcal{B}_i,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \, \mathrm{d}\mathcal{H}^2 = 0,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = \sqrt{\mathcal{R}_2(\xi_i)} \frac{\mathrm{d}B_i}{\mathrm{d}t},$$

$$X_i(t) = X_i(0), \quad \mathrm{d}\xi_i = \xi_i \times \Omega_i.$$

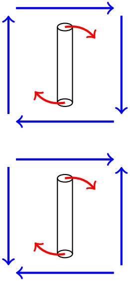
Inertialess rodlike particles

For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + \cancel{u \cdot \nabla_x f} + \cancel{\mathrm{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f)} = \cancel{D_t \Delta_x f} + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \mathrm{div}\, \sigma := \mathrm{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \mathrm{div}\, u = 0.$$

Analyzed by Helzel, Tzavaras and Otto.



Ongoing work with Amina Mecherbet and Marta Leocata: Derive the onset of the stress σ from a microscopic model.

$$-\Delta u_N + \nabla p_N = 0, \quad \mathrm{div}\, u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_i \mathcal{B}_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } \mathcal{B}_i,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \, \mathrm{d}\mathcal{H}^2 = 0,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = \sqrt{\mathcal{R}_2(\xi_i)} \frac{\mathrm{d}B_i}{\mathrm{d}t},$$

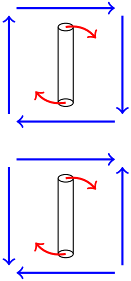
$$X_i(t) = X_i(0), \quad \mathrm{d}\xi_i = \xi_i \times \mathrm{d}B_i.$$

Inertialess rodlike particles

For inertialess rodlike Brownian particles (omitting gravity), Doi '81 proposed

$$\partial_t f + \cancel{u \cdot \nabla_x f} + \cancel{\operatorname{div}_\xi (P_{\xi^\perp}(\xi \cdot \nabla u) f)} = \cancel{D_t \Delta_x f} + D_r \Delta_\xi f,$$

$$-\Delta u + \nabla p = \operatorname{div} \sigma := \operatorname{div} \int_{S^2} (3\xi \otimes \xi - \mathrm{Id}) f \, \mathrm{d}\xi, \quad \operatorname{div} u = 0.$$



Analyzed by Helzel, Tzavaras and Otto.

Ongoing work with Amina Mecherbet and Marta Leocata: Derive the onset of the stress σ from a microscopic model.

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_i \mathcal{B}_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } \mathcal{B}_i,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \, \mathrm{d}\mathcal{H}^2 = 0,$$

$$\int_{\partial \mathcal{B}_i} \sigma[u_n, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = \sqrt{\mathcal{R}_2(\xi_i)} \frac{\mathrm{d}B_i}{\mathrm{d}t},$$

$$X_i(t) = X_i(0), \quad \mathrm{d}\xi_i = \xi_i \times \mathrm{d}B_i.$$

Thank you!