Derivation of sedimentation models as a mean-field limit and analysis of the transport-Stokes model

Collective behavior of particles in fluids

Amina MECHERBET

Université de Paris

IMJ-PRG

Outline

1. On mean-field limits
2. On the sedimentation of a falling droplet
   - Global existence and uniqueness result for (TS)
   - Evolution of the surface of the droplet
   - Investigation of the spherical shape case

Microscopic model for inertialess sedimentation

n particles B_i centered in x_i with orientation θ_i and radius $R = \frac{r_0}{n}$, $1 \leq i \leq n$

$$\begin{cases} -\Delta u + \nabla p = 0, \operatorname{div}(u) = 0, & \text{on } \mathbb{R}^3 \setminus \bigcup_{i=1}^{n} \bar{B}_i \\ u = u_i + \omega_i \times (x - x_i), & \text{on } B_i, 1 \leq i \leq n, \\ \lim_{|x| \to \infty} |u(x)| = 0. & \end{cases}$$

$$\int_{\partial B_i} \Sigma(u,p)\nu d\sigma(x) + \frac{4}{3}\pi r^3(\rho_p - \rho_f)g = 0, \quad \int_{\partial B_i} (x - x_i) \times [\Sigma(u,p)\nu] d\sigma(x) = 0,$$

$$\dot{x}_i = u_i, \quad \dot{\theta}_i = \omega_i.$$

System of interacting spherical particles

As long as $d_{\min} \geq \frac{c}{n}$ and (roughly) $r_0 = Rn$ small enough we have

$$\dot{x}_i = \kappa g + \frac{6\pi r_0}{n} \sum_{j \neq i} \Phi(x_i - x_j)\kappa g + O(d_{\min})$$

- $\Phi(x) = \frac{1}{8\pi}\left(\frac{I}{|x|} + \frac{x \otimes x}{|x|^3}\right)$ the Oseen tensor
- $\kappa g = \frac{2}{9} R^2(\rho_p - \rho_f) g$ the velocity fall of one single spherical particle in a Stokes flow

- P. E. Jabin and F. Otto (2004). R. M. Höfer (2018). A. M (2019). R. M. Höfer and R. Schubert (2020).

Propagation of the infinite Wasserstein distance and the minimal inter-particle distance

$$\mathcal{K}^n\rho^n(x) = 6\pi r_0 \int_{\mathbb{R}^3} \chi\Phi(x-y)\kappa g \rho^n(dy), \quad \mathcal{K}\rho(x) = 6\pi r_0 \int_{\mathbb{R}^3} \Phi(x-y)\kappa g \rho(dy)$$

with $\rho^n = \frac{1}{n}\sum_i \delta_{x_i}$, formally we have

$$\frac{d}{dt}W_\infty(\rho^n,\rho) \lesssim \|\mathcal{K}^n\rho^n - \mathcal{K}\rho\|_\infty, \qquad \frac{d}{dt}d_{\min} \lesssim d_{\min}\|\nabla\mathcal{K}^n\rho^n\|_\infty$$

$$\|\mathcal{K}^n\rho^n - \mathcal{K}\rho\|_\infty \lesssim W_\infty\left(1 + W_\infty + \frac{W_\infty^2}{d_{\min}}\right), \quad \|\nabla\mathcal{K}^n\rho^n\|_\infty \lesssim \left(1 + \frac{W_\infty^3}{d_{\min}^2}\right)$$

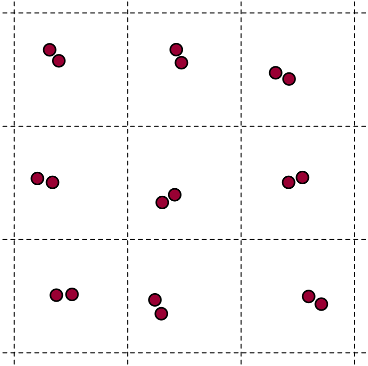
Cluster configuration

As a first step for rigorous justification of polymeric suspensions

n pairs of spheres centered in x_1^i, x_2^i

$$|x_1^i - x_2^i| \sim R, 1 \leq i \leq n.$$

$x_+^i := \frac{x_1^i + x_2^i}{2}$ center of the i^{th} cluster
$\xi^i := \frac{x_1^i - x_2^i}{2R}$ orientation of the i^{th} cluster



- C. Le Bris, T. Lelièvre. Micro-macro models for viscoelastic fluids: modelling, mathematics and numerics 2011.

System of interacting clusters

As long as $d_{\min} \geq \frac{c}{\sqrt{n}}$ and r_0 small enough we have

$$\dot{x}^i_+ = (\mathbb{A}(\xi_i))^{-1} \kappa g + \frac{6\pi r_0}{n} \sum_{j \neq i} \Phi(x^i_+ - x^j_+) \kappa g + O(d_{\min})$$

$$\dot{\xi}_i = \left(\frac{6\pi r_0}{n} \sum_{j \neq i} \nabla \Phi(x^i_+ - x^j_+) \kappa g \right) \cdot \xi_i + O(d_{\min})$$

- $(\mathbb{A}(\xi_i))^{-1} \kappa g$ the mean velocity fall of one pair of identical spheres in a Stokes flow

- D.J. Jeffrey and Y. Onishi, Calculation of the resistance and mobility functions for two unequal spheres in low-Reynolds-number flow. J. Fluid Mech. 139 (1984) 261–290.
- A. M. A model for suspension of clusters of particle pairs. 2019. hal-02171615. To appear in ESAIM: Mathematical Modelling and Numerical Analysis.

System of interacting axisymmetric particles

In the case where the shape has 3 axes of symmetry the system obtained is

$$\dot{x}_i \sim (\mathcal{M}_1(\xi_i))\,\kappa g + \frac{6\pi r_0}{n}\sum_{j\neq i}\Phi(x_+^i - x_+^j)\kappa g$$

$$\dot{\xi}_i \sim \mathcal{M}_2(\xi_i)\left(\frac{6\pi r_0}{n}\sum_{j\neq i}\nabla\Phi(x_+^i - x_+^j)\kappa g\cdot\xi_i\right)$$

- $(\mathcal{M}_1(\xi_i))\,\kappa g$ the velocity fall of one single particle in a Stokes flow
- (x_i, ξ_i) the center and orientation of the i^{th} particle

Control of the minimal distance

Cluster configuration

Proposition

For all $1 \leq i \leq N$ *and* $j \neq i$ *we have*

$$\begin{aligned} |\dot{\xi}_i| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty |\xi_i| + O(d_{\min}), \\ \left|\dot{x}^i_+ - \dot{x}^j_+\right| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty |x^i_+ - x^j_+| + |\xi_i - \xi_j| + O(d_{\min}), \\ \left|\dot{\xi}_i - \dot{\xi}_j\right| &\lesssim \|\nabla \mathcal{K}^N \rho^N\|_\infty \left|\xi_i - \xi_j\right| + \|\nabla^2 \mathcal{K}^N \rho^N\|_\infty \left|x^i_+ - x^j_+\right| + O(d_{\min}). \end{aligned}$$

Proposition

There exists a positive constant $C > 0$ *independent of* N *such that:*

$$\|\mathcal{K}^N \rho^N\|_{W^{2,\infty}(\mathbb{R}^3)} \leq C \left(\frac{W_\infty^3}{d_{\min}} + \frac{W_\infty^3}{d_{\min}^2} + \frac{W_\infty^3}{d_{\min}^3} \right) \|\rho\|_{W^{1,\infty}(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)}.$$

Limitation of the mean-field approach

Cluster configuration

Formally

$$\frac{d}{dt}W_\infty(\rho^N(t,\cdot),\rho(t,\cdot)) \lesssim \|\mathcal{K}^N\rho^N - \mathcal{K}\rho\|_{L^\infty} + \|\nabla\mathcal{K}^N\rho^N - \nabla\mathcal{K}\rho\|_{L^\infty}$$

$$\begin{aligned}
\|\mathcal{K}^N\rho^N - \mathcal{K}\rho\|_{L^\infty} &\lesssim \|\rho\|_\infty W_\infty(\rho^N,\rho)\left(1 + \frac{W_\infty(\rho^N,\rho)^2}{d_{\min}}\right) \\
\|\nabla\mathcal{K}^N\rho^N - \nabla\mathcal{K}\rho\|_{L^\infty} &\lesssim \|\rho\|_\infty W_\infty(\rho^N,\rho)\Big(|\log W_\infty(\rho^N,\rho)| \\
&+ \frac{W_\infty(\rho^N,\rho)^2}{d_{\min}^2} + 1\Big)
\end{aligned}$$

Outline

1. On mean-field limits

2. On the sedimentation of a falling droplet
   - Global existence and uniqueness result for (TS)
   - Evolution of the surface of the droplet
   - Investigation of the spherical shape case

A mesoscopic model for inertialess sedimentation

$(t,x) \mapsto (u(t,x), p(t,x))$ the fluid velocity and pressure.

$(t,x) \mapsto \rho(t,x)$ the (probability) density of the suspension.

$$\begin{cases} \partial_t \rho + \mathrm{div}((u+\kappa g)\rho) &= 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ -\Delta u + \nabla p &= 6\pi r_0 \kappa \rho g\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathrm{div}\, u &= 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ u &= 0, & \text{at infinity} \\ \rho(0,\cdot) &= \rho_0\,, & \text{on } \mathbb{R}^3. \end{cases} \tag{TS}$$

$\kappa g = \frac{2}{9} r^2 (\rho_p - \rho_f) g$ the velocity fall of one particle given by Stokes law.

$6\pi r_0 \kappa g = n \frac{4}{3} \pi r^3 (\rho_p - \rho_f) g \rho := \lambda (\rho_p - \rho_f) g \rho$ the Brinkman force or the collective force applied by the suspension on the fluid where λ the volume fraction of the suspension.

Experimental and numerical behaviour of falling dispersed particles/droplets in a viscous fluid

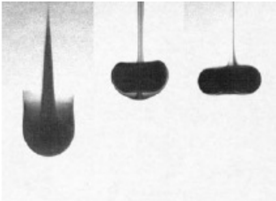


Figure: Droplet breakup and torus formation[1]

- G. K. Batchelor and J. M. Nitsche, *Break-up of a falling drop containing dispersed particles*, J . Fluid Mech. Volume 340, (10 June 1997)
- [1] G. Machu, W. Meile, L. C. Nitsche, and U. Schaflinger, *Coalescence, torus formation and breakup of sedimenting drops: experiments and computer simulations*, J . Fluid Mech. Volume 447, (25 November 2001)
- B. Metzger, M. Nicolas, and É. Guazzelli, *Falling clouds of particles in viscous fluids*, J . Fluid Mech. Volume 580, (10 June 2007), pp. [283,301].

Comparison to the falling of a liquid viscous droplet in a lighter viscous fluid

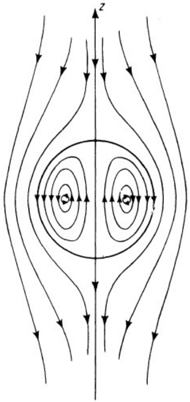


Figure: Streamlines for liquid droplet showing internal circulation[1]

Invariance of the spherical shape falling according to a translational velocity given by

$$V = \frac{2}{9\mu} r^2 (\rho_l - \rho) g \frac{\mu_l + \mu}{\mu_l + \frac{2}{3}\mu},$$

μ_l, ρ_l the viscosity and density of the interior liquid in the sphere.

- M. J. HADAMARD, *Mouvement permanent lent d'une sphere liquide et visqueuse dans un liquide visqueux*, C. R. Acad. Sci. 152, (1911)
- [1] J. HAPPEL AND H. BRENNER Low Reynolds number hydrodynamics. Figure 4-21.1. 1965
- W. RYBCZYNSKI, *Über die fortschreitende bewegung einer flüssigen kugel in einem zähen medium*, Bull. Acad. Sci. Cracovie, (1911)

Analysis of the TS model

Global existence and uniqueness result for regular initial densities can be found in the paper of R. M. Höfer.
Up to a change of variable, we consider the following equation in the following

$$
\begin{cases}
\partial_t \rho + \mathrm{div}(\rho u) & = & 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
-\Delta u + \nabla p & = & -\rho e_3\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\mathrm{div}\, u & = & 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
u & = & 0, & \text{at infinity} \\
\rho(0,\cdot) & = & \rho_0\,, & \text{on } \mathbb{R}^3.
\end{cases}
\tag{1}
$$

- R. M. Höfer 2018, R. M. Höfer and R. Schubert 2020.

Outline

1. On mean-field limits

2. On the sedimentation of a falling droplet
   - Global existence and uniqueness result for (TS)
   - Evolution of the surface of the droplet
   - Investigation of the spherical shape case

Theorem (AM 20')

Let $\rho_0 \in L^\infty(\mathbb{R}^3)$ *a probability measure with finite first moment. There exits a unique couple* $(\rho, u) \in L^\infty(0, T; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \times L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3))$ *satisfying the transport-Stokes equation* (1) *for all* $T \geq 0$*. Moreover, for all* $s \in [0, T]$ *there exists a unique characteristic flow* $X(\cdot, s, \cdot) \in L^\infty(0, T, W^{1,\infty}(\mathbb{R}^3))$

$$\begin{cases} \partial_t X(t,s,x) &= u(s, X(t,s,x)), & \forall\, t, s \in [0, T], \\ X(s,s,x) &= x, & \forall\, s \in [0, T], \end{cases}$$

For all $s, t \in [0, T]$ *the diffeomorphism* $X(s, t, \cdot)$ *is measure preserving and we have*

$$\rho(t, \cdot) = X(t, 0, \cdot)\#\rho_0.$$

Regularity of the velocity field

Proposition

Let $\eta \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, the unique u solution to the Stokes equation:

$$\begin{cases} -\Delta u + \nabla p &= -\eta e_3\,, \quad \text{on } \mathbb{R}^3 \\ \operatorname{div}(u) &= 0\,, \quad \text{on } \mathbb{R}^3, \\ u &= 0, \quad \text{at infinity} \end{cases}$$

is given by $u = -\Phi \star \eta e_3$ with the Oseen tensor Φ

$$\Phi(x) = \frac{1}{8\pi}\left(\frac{\mathbb{I}_3}{|x|} + \frac{x \otimes x}{|x|^3}\right).$$

$u \in W^{1,\infty}(\mathbb{R}^3)$ and there exists a positive constant independent of the data such that:

$$\|u\|_\infty + \|\nabla u\|_\infty \leq C\|\eta\|_{L^1 \cap L^\infty}.$$

Stability estimates using the first Wasserstein distance

For any $\eta_1, \eta_2 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $u_i = -\Phi \star \eta_i e_3$ we have

$$\int_{\mathbb{R}^3} |u_1(x) - u_2(x)| \rho(dx) \leq C(\|\eta_i\|_{L^1 \cap L^\infty}) W_1(\eta_1, \eta_2).$$

For any $u_i \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3))$, $i = 1, 2$ and ρ_i the solution to the associated transport equation

$$W_1(\rho_1(t), \rho_2(t)) \leq \left(W_1(\rho_1(s), \rho_2(s)) + \int_s^t \int_{\mathbb{R}^3} |u_2(\tau, x) - u_1(\tau, x)| \, \rho_1(\tau, x) dx d\tau \right) e^{Q_2(t-s)},$$

where $Q_i := \|u_i\|_{L^\infty(0,T;W^{1,\infty})}$.

- M. Hauray and P. E. Jabin, *Particle approximation of Vlasov equations with singular forces : propagation of chaos*, Ann. Sci. Éc. Norm. Supér. (4), (2015).

Outline

1. On mean-field limits
2. On the sedimentation of a falling droplet
   - Global existence and uniqueness result for (TS)
   - Evolution of the surface of the droplet
   - Investigation of the spherical shape case

Derivation of an equation for the surface evolution

If $\rho_0 = 1_{B_0}$ then $\rho_t = 1_{B_t}$ where B_t is transported along the flow.
Let consider

$$\partial B_0 = \left\{ r_0(\theta) \begin{pmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\},$$

We set then $c(t) = (0, 0, c_3(t)) \in B_t$ the position at time t of a reference point such that $c(0) = 0$ and write $B_t = c(t) + \tilde{B}_t$ where

$$\partial \tilde{B}_t = \left\{ r(t, \theta) \begin{pmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}.$$

Derivation of the evolution equation for the surface droplet

$$\begin{cases} \partial_t r + A_1[r]\partial_\theta r &= A_2[r], \quad \text{on } \mathbb{R}^+ \times [0,\pi] \\ r(0,\cdot) &= r_0, \quad \text{on } [0,\pi] \end{cases} \tag{H}$$

In the case where the reference point $c = (0,0,c_3)$ is transported along the flow *i.e.* $u(c) = \dot{c}$ we have $c = c[r] = (0,0,c[r]_3)$ and

$$\begin{cases} \dot{c}[r]_3(t) &= -\frac{1}{4}\int_0^\pi r^2(t,\bar{\theta})\sin(\bar{\theta})\left(1 - \dfrac{1}{2}\sin^2(\bar{\theta})\right)d\bar{\theta}, \\ c[r]_3(0) &= 0, \end{cases} \tag{C}$$

Remark

The volume of the droplet is conserved in time

$$\int_0^\pi \partial_t r(t,\theta) r^2(t,\theta)\sin(\theta)d\theta = 0.$$

$$A_1[r] = \frac{1}{r}(\mathcal{U}[r] - \dot{c}) \cdot \partial_\theta e(\cdot, 0), \quad A_2[r] = (\mathcal{U}[r] - \dot{c}) \cdot e(\cdot, 0).$$

$$\mathcal{U}[r](\theta) = \int_{(0,\pi)\times(0,2\pi)} \int_0^{r(\bar\theta)} \Phi(r(\theta)e(\theta,0) - ze(\bar\theta,\bar\phi))z^2 \sin(\bar\theta) dz d\bar\theta d\bar\phi,$$

$$\begin{aligned}\mathcal{U}[r](\theta) = -\frac{1}{8\pi}\int_{[0,\pi]\times[0,2\pi]} \Bigg(& \frac{\left(r(\theta)e(\theta,0) - r(\bar\theta)e(\bar\theta,\bar\phi)\right)\cdot e_3}{\left|r(\theta)e(\theta,0) - r(\bar\theta)e(\bar\theta,\bar\phi)\right|} s[r](\bar\theta,\bar\phi) \\ & - \frac{\left(r(t,\theta)e(\theta,0) - r(\bar\theta)e(\bar\theta,\bar\phi)\right)\cdot s[r](\bar\theta,\bar\phi)}{\left|r(\theta)e(\theta,0) - r(\bar\theta)e(\bar\theta,\bar\phi)\right|} e_3 \Bigg) d\bar\theta d\bar\phi.\end{aligned}$$

$e(\theta,\phi) \in \mathbb{S}^2$, $s[r](\bar\theta,\bar\phi)$ the surface element on $\tilde{B}_t$.

$$A_1[r](t,\theta) :=$$

$$-\frac{1}{8\pi r(t,\theta)}\int_0^{2\pi}\int_0^{\pi}\frac{r(t,\bar\theta)\sin(\bar\theta)-\partial_\theta r(t,\bar\theta)\cos(\bar\theta)}{\beta[r](t,\theta,\bar\theta,\phi)}r(t,\bar\theta)\sin(\bar\theta)\Big(r(t,\theta)\cos(\phi) - r(t,\bar\theta)\Big\{\cos(\bar\theta)\cos(\theta)\cos(\phi)+\sin(\bar\theta)\sin(\theta)\Big\}\Big)d\bar\theta d\phi + \frac{\dot{c}_3\sin(\theta)}{r(t,\theta)}$$

$$A_2[r](t,\theta) :=$$

$$-\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi}\frac{r(t,\bar\theta)\sin(\bar\theta)-\partial_\theta r(t,\bar\theta)\cos(\bar\theta)}{\beta[r](t,\theta,\bar\theta,\phi)}r(t,\bar\theta)\sin(\bar\theta)\Big(-r(t,\bar\theta)\sin(\theta)\cos(\bar\theta)\cos(\phi) + r(t,\bar\theta)\cos(\theta)\sin(\bar\theta)\Big)d\bar\theta d\phi - \dot{c}_3 cos(\theta)\,.$$

$$\beta[r](\theta,\bar\theta,\phi)^2 = r^2(\theta)+r^2(\bar\theta)-2r(\theta)r(\bar\theta)(\sin(\theta)\sin(\bar\theta)\cos(\phi)+\cos(\theta)\cos(\bar\theta)).$$

Estimates on the non local operators

We set

$$|r|_* = \inf_{(0,\pi)} r(\theta) > 0.$$

$$|\mathcal{U}[r](\theta)| \leq C \int_{(0,\pi)\times(0,2\pi)} \int_0^{r(\bar{\theta})} \frac{z^2 dz}{|r(\theta)e(\theta,0) - ze(\bar{\theta},\bar{\phi})|} \sin(\bar{\theta}) d\bar{\theta} d\bar{\phi},$$

$$\leq \frac{\|r\|_\infty^{5/2}}{\sqrt{|r|_*}} \int_0^{2\pi} \int_0^{\pi} \frac{\sin(\bar{\theta}) d\bar{\theta} d\bar{\phi}}{|e(\bar{\theta},\bar{\phi}) - e(\theta,0)|}$$

$$\int_0^{2\pi} \int_0^{\pi} \frac{\sin(\bar{\theta}) d\bar{\theta} d\bar{\phi}}{|e(\bar{\theta},\bar{\phi}) - e(\theta,0)|} = \int_{\partial B(0,1)} \frac{d\sigma(y)}{|y-x|} < C, \forall\, x \in \partial B(0,1)$$

bounds and stability estimates

$$
\begin{aligned}
\|A_1[r]\|_{1,\infty} &\leq K\left(\|r\|_{1,\infty}, \frac{1}{|r|_*}\right) \\
\|A_1[r_1] - A_1[r_2]\|_\infty &\leq K\left(\frac{1}{|r_1|_*}, \frac{1}{|r_2|_*}, \|r_1\|_\infty, \|r_2\|_\infty\right) \|r_1 - r_2\|_\infty
\end{aligned}
$$

Remark

$$
A_1[r](t,0) = A_1[r](t,\pi) = 0.
$$

The characteristic curves are well defined

Local existence and uniqueness

Theorem (AM 20')

Let $r_0 \in \mathcal{C}^{0,1}[0,\pi]$ such that $|r_0|_* > 0$. There exists $T > 0$ and a unique $r \in \mathcal{C}(0, T; \mathcal{C}^{0,1}(0,\pi))$ satisfying the hyperbolic equation (H)*. Moreover, there exists a unique associated reference point $c = c[r] \in \mathcal{C}(0, T)$ satisfying* (C) *.*

Remark

The same result holds true if the motion of the center c is defined in another way. The only properties needed is a uniform bound on $\dot{c}$ and a stability estimate with respect to r if $c = c[r]$.

Outline

1. On mean-field limits

2. On the sedimentation of a falling droplet
   - Global existence and uniqueness result for (TS)
   - Evolution of the surface of the droplet
   - Investigation of the spherical shape case

Recovering the Hadamard and Rybczynksi result in the spherical case $B_0 = B(0,1)$ means that we should have $B_t = v^* t + B_0$ wtih v_* the velocity fall of the droplet given by

$$v^* = \frac{2}{9}\frac{R^2}{\mu}(\bar{\rho} - \rho)\frac{\mu + \bar{\mu}}{\bar{\mu} + \frac{2}{3}\mu}g. \tag{2}$$

In particular, we can recover explicitly the following property showed by Hadamard and Rybczynski

Lemma (Hadamard–Rybczynski)

Let $u_0 = -\Phi * 1_{B_0} e_3$, $v^* = -\frac{4}{15}e_3$. We have

$$(u_0 - v^*) \cdot n = 0 \text{ on } \partial B_0$$

Corollary

The solution (u, ρ) *of the transport-Stokes equation* (1) *in the case where* $\rho_0 = 1_{B_0}$ *is given by*

$$u(t,x) = u_0(x - v^* t), \quad \rho(t,x) = \rho_0(x - v^* t),$$
$$u_0 = -\Phi * \rho_0 e_3, \qquad \rho_0 = 1_{B(0,1)}.$$

In other words, the drop B_t *remains spherical for all time.*

Proof.

$$\partial_t \rho + \nabla \rho \cdot u = (\nabla \rho_0 \cdot (u_0 - v^*))_{|(\cdot - v^* t)} = 0,$$

we conclude using the fact that $\nabla \rho_0 = n s^1$ where s^1 is the surface measure on the sphere and n the unit normal. □

Recovering the spherical shape in the hyperbolic equation $\partial_t r + A_1[r]\partial_\theta r = A_2[r]$

Case 1. We set $c = v^* t$. The result is straightforward.

Proof.

We recall that

$$A_2[r] = (\mathcal{U}[r] - \dot{c}) \cdot e(\cdot, 0)$$

At $t = 0$, $r = r_0 = 1$ and $A_2[1] = (u_0 - v^*) \cdot n = 0$ hence $r_0 = 1$ is solution all the time. □

Case 2. $c \neq v^* t$

Proposition

Let $r_0 = 1$ *and* (r, c) *the solution of* (H), *with* $\dot{c} \neq v^*$. *Denote by* $T > 0$ *the maximal time of existence of the solution such that* $|c - c^*| \leq 1$ *with* $c^* = v^* t = -\frac{4}{15} e_3 t$. *Then* r *is given by*

$$r(t, \theta) = -(c - c^*)_3 \cos(\theta) + \sqrt{1 - (c - c^*)_3^2 \sin^2(\theta)}, \ (t, \theta) \in [0, T] \times [0, \pi]$$

and satisfies

$$|c(t) + r(t, \theta) e(\theta, 0) - v^* t|^2 = 1 \ \text{for all} \ \theta \in [0, \pi] \ \text{and} \ t \leq T.$$

In other words

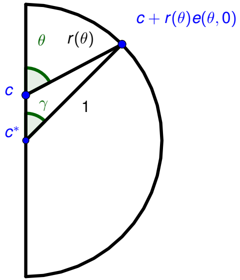
$$\partial B_t := c + \partial \tilde{B}_t = \partial B(c^*, 1) \ \text{on} \ [0, T].$$

Proof.

$$\bar{r}(t,\theta) = -(c-c^*)_3\cos(\theta) + \sqrt{1-(c-c^*)_3^2\sin^2(\theta)},\ (t,\theta)\in[0,T]\times[0,\pi]$$

is a solution iff

$$(\dot{c}^* - \mathcal{U}[\bar{r}])\cdot(\bar{r}e(\theta,0)+c-c^*) = 0$$



Use the change of variable

$$c + \bar{r}(\theta)e(\theta,0) - c^* = e(\gamma,0)$$

and show that

$$\mathcal{U}[\bar{r}](t,\theta) = \mathcal{U}[1](\gamma)$$

□

Global existence in the spherical case

We assume that c is transported along the flow

$$\begin{cases} \dot{c}[r]_3(t) &= -\frac{1}{4}\int_0^\pi r^2(t,\bar{\theta})\sin(\bar{\theta})\left(1-\frac{1}{2}\sin^2(\bar{\theta})\right)d\bar{\theta}, \\ c[r]_3(0) &= 0, \end{cases} \tag{C}$$

Proposition

Let $r_0 = 1$ *and* (r, c) *the solution of* (H) *and* (C). *For all time* $t \geq 0$ *we have* $c_3(t) \leq c_3^*(t)$, $|c(t) - c^*(t)| \leq 1$ *and*

$$\lim_{t\to\infty} c_3(t) - c_3^*(t) = -1$$

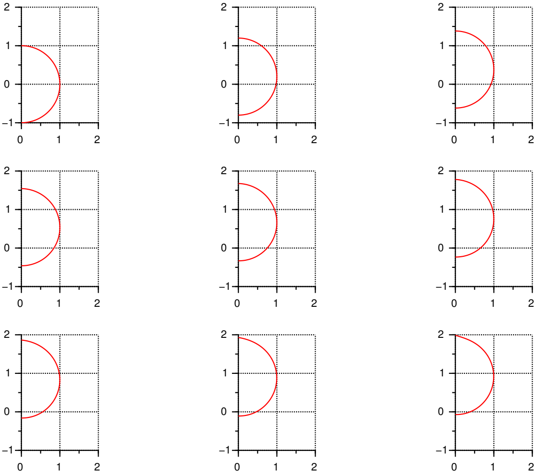


Figure: Droplet evolution for $t = 0, 3, \cdots, 24$

Numerical simulation using upwind finite difference scheme

Numerical investigation of a special case

We choose $\dot{c} = \lambda \dot{c}^*$ with $\lambda > 1$. We have

$$|c(t) - c^*(t)| = t(\lambda - 1)|v^*| = t(\lambda - 1)\frac{4}{15},$$

if we set for instance $\lambda = \frac{17}{2}$, the time $\bar{t}$ for which one should have $|c(\bar{t}) - c^*(\bar{t})| = 1$ is $\bar{t} = 0.5$.

t	0	0.1	0.2	0.3	0.35	0.4	0.45	0.49	0.5
$\|c - c^*\|$	0.02	0.22	0.42	0.62	0.72	0.82	0.92	1.00	1.02
$E_1^n(\times.10^{-2})$	0.02	0.22	0.48	0.83	1.08	1.4	1.86	2.51	2.82
$\min_i r_i^n$	0.98	0.78	0.58	0.38	0.28	0.1805	0.0807	0.0009	-0.1394
$V^n(\times.10^{-2})$	0.03	0.436	0.87	1.38	1.68	2.014	2.42	2.84	-

Table: Second test case. Evolution of E_1^n, $\min_i r_i^n$ and V^n

Ongoing work

- Investigation of an appropriate scheme for the hyperbolic equation (convergence result and stability, ensuring the steady state approximation?)
- Investigation of other axisymmetric shapes such as ellipsoids

$$r_0(\theta) = \frac{1}{\sqrt{1 - \frac{3}{4}\cos^2(\theta)}}, \quad r_0(\theta) = \frac{1}{\sqrt{1 - \frac{3}{4}\sin^2(\theta)}}, \quad \theta \in [0, \pi],$$

“Simulation” of an ellipse case